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Particle Production in 5-Dimensional Cosmological Relativity

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The 5-dimensional extension of cosmological special and general relativity is considered. In this framework it is possible to define a 5-dimensional perfect fluid stress-energy tensor and to unify the equations of perfect hydrodynamics in a single 5-dimensional tensor conservation law. This picture in principle permits to interpret particle production phenomena as cosmological effects, in the spirit of open system cosmology.

KEY WORDS: cosmological relativity; open system cosmology; relativistic hydrodynamics.

1. INTRODUCTION

Recently it was noticed that the constancy of the expansion of the Universe, according to Hubble's law, in the limit of negligible gravitational force, can be cast in a framework analogue to that of Einstein's special relativity, where the receding velocity v of galaxies takes the place of time, and universal observers at a fixed cosmic time take the place of moving frames, giving rise to so-called cosmological special relativity (Carmeli, 1995, 1996). In fact assuming that the expansion of the Universe satisfies Hubble's law at different cosmic times t and t', we have:

$$-\tau^2 v^2 + (x^2 + y^2 + z^2) = -\tau^2 v'^2 + (x'^2 + y'^2 + z'^2) = 0, \qquad (1)$$

where $\tau = 1/H_0$ is Hubble's time. Convention is adopted that at present we have $t = 0$ and that cosmological time is counted backwards. Thus one finds that observers at different cosmic times are related each other by cosmological transformations in a Minkowskian space-velocity, analogue to Lorentz transformations in space-time. In particular the null cone of the space-velocity, called galaxy cone, is invariant and represents the location of the galaxies in the expanding universe.

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Many interesting cosmological effects, analogue to well known special relativistic effects, directly follow from the model, such as lenght contraction and velocity contraction. A self contained overview of the theory and its implications, including a picture of inflation and the possibility that the dark matter hypothesis could be abolished, can be found in Carmeli (2002). The mass-energy content of the Universe is actually considered in the curved space-velocity extension of cosmological special relativity, called cosmological general relativity (see Carmeli, 2002, Appendix A), where gravitational force is not neglected. The re-introduction of time as the fifth dimension then leads to so-called 5-dimensional brane world theory (see Carmeli, 2002, Appendix B).

However, in the limit of negligible gravitation, it is worth to consider special relativistic dynamics (of "test" mass-energy distributions), where it is possible in fact to attempt a direct 5-d extension of cosmological special relativity. Actually in this paper we consider 5-d cosmological special relativistic hydrodynamcs and show that this leads in a natural way to particle production effects. The picture does not change significantly when adding gravitation on a 5-d curved manifold (with the presence of a 5-d self-gravitating cosmological fluid); thus particle production seems to be interpretable as an effect due to the fifth dimension (velocity, or time, depending on the point of view).

Recent relativistic inflationary cosmology often considers particle production phenomena (see e.g. Cissoko, 1998). In fact cosmological particle production accounts for negative pressure, which arises in the modeling of the accelerating universe (de Campos, 2002). The cosmological scenario with particle production is usually called open system cosmology (Prigogine *et al.*, 1989). Here we see that such scenario can fit in a natural way in the framework of 5-d cosmological relativity. However the effect here seems to be independent from the sign of the pressure of the cosmological fluid.

The differential system of special relativistic hydrodynamics (see Section 2) is a set of five scalar equations: four come from the stress-energy tensor conservation $\partial_\alpha T^{\alpha\beta} = 0$; the fifth is the scalar conservation law $\partial_\alpha(rU^\alpha) = 0$, which corresponds to the number density conservation ($\alpha, \beta = 0, 1, 2, 3$).

It is then reasonable to imagine their unification into a single 5-d tensor conservation law: $\partial_A T^{AB} = 0$ ($A, B = 0, 1, 2, 3, 4$; see Section 3). This however lets some additional terms, due to the fifth dimension, arise (Section 4), which can be interpreted in the light of open system cosmology. In fact these terms turn the stress-energy and the number density conservation laws into balance laws; this seems to suggest that the fifth dimension plays a role in problems, such as relativistic inflationary cosmology, where the barion number is not conserved (Section 5).

The famous Kaluza–Klein unified theory of gravity and electromagnetism is the prototype 5-d extension of general relativity. One of the aims of this theory is however to show how 5-d vacuum turns into 4-d stress-energy; rather, here instead we have that 5-d matter conservation turns into 4-d particle creation. Also Kaluza–Klein inflationary scenarios considered in the literature have 5-d vacuum turning into 4-d matter and entropy generation. Here we follow a different approach: 4-d stress-energy is simply obtained from its 5-d analogue by projection, and similarly for the 4-d Einstein tensor.

Some extensions of Kaluza–Klein theory admit a suitable 5-d stress-energy tensor for compactification reasons (for a recent overview of Kaluza–Klein theory of gravity and its developments see Overduin and Wesson, 1998), but a general 5-d stress-energy tensor does not usually fit in 5-d general relativity. This is a reason for considering 5-d hydrodynamics first in the simpler framework of an extended special relativity. When enlarging the framework to the same extension of general relativity we find out however that the main features of the hydrodinamical equations are the same. This is not, of course, for the link between dynamics (including particle production) and the geometry of the space-time, which in the former case is absent, while in the latter case is governed by the 5-d Einstein equations.

Density and stress-energy non-conservation however appears to be more a feature of 5-d dynamics rather than of Kaluza–Klein or brane world geometry. In fact it can be cast in the framework of 5-d special relativity, as well as in 5-d general relativity. Even in the simpler framework the fifth dimension is still interpretable as the galaxy receding velocity, thus particle production appears to be a genuine cosmological effect.

2. RELATIVISTIC HYDRODYNAMICS

Let $\mathcal{M}_4$ denote the minkowski space-time, of signature $-+++$. Let greek indices run from 0 to 3. Units are chosen in order to have the speed of light in empty space $c \equiv 1$.

A relativistic continuum system is characterized by a world tube $\Omega \subset \mathcal{M}_4$, generated by the set of the worldlines of its constituting elementary particles, with tangent unit timelike vector U^α. The world tube Ω is the support of a stress-energy tensor $T^{\alpha\beta}$, which satisfy the conservation law: $\partial_\alpha T^{\alpha\beta} = 0$.

For a perfect fluid the matter-energy tensor is a function of the dynamical and thermodynamical variables (see for ex. Anile, 1989; Ferrarese, 1985; Lichnerowicz, 1967, 1994):

$$T^{\alpha\beta} = (\rho + p)U^\alpha U^\beta + p\eta^{\alpha\beta} \tag{2}$$

where $\eta^{\alpha\beta}$ is the Minkowski metric, $\rho \geq 0$ is the proper energy density and $p \geq 0$ the proper pressure. We suppose $\rho + p > 0$. We moreover set

$$\rho = r(1 + \mathcal{E}) \tag{3}$$

where $r \geq 0$ is the matter density (barion number) and $\mathcal{E} \geq 0$ the internal energy. Let us introduce the proper temperature T, the specific entropy S and the thermodynamic principle

$$T\,dS = d\mathcal{E} - \frac{p}{r^2}dr \tag{4}$$

It is convenient to adopt p and S as the fundamental thermodynamical variables, and to introduce a generic equation of state of the form: $r = r(p, S)$. Thus from (4) we have two independent relations: $T = (\partial\mathcal{E}/\partial S) - pr^{-2}(\partial r/dS)$ and $T(\partial S/\partial p) = (\partial\mathcal{E}/\partial p) - pr^{-2}(\partial r/dp)$. We therefore have 5 independent variables: p, S (or any other pair of thermodynamical variables) and the three independent components of U^α ($U^\alpha U_\alpha = -1$).

The differential system of relativistic hydrodynamics is the following:

$$\partial_\alpha(rU^\alpha) = 0, \qquad \partial_\alpha T^{\alpha\beta} = 0 \tag{5}$$

Here $(5)_1$ means conservation of the matter density (which corresponds to the conservation of the specific number of particles); therefore nuclear reactions of particle creation are neglected within this scheme. As for $(5)_2$, it is the ordinary conservation equation for the energy tensor of the fluid. The total number of scalar equations of (5) is 5.

An equivalent formulation of (5) is obtained by replacing equation $\partial_\alpha(rU^\alpha) = 0$ with $U^\alpha\partial_\alpha S = 0$. However formulation (5) is preferable for its conservative form.

In the case of general relativity the Minkowski spacetime $\mathcal{M}_4$ is replaced by a generic space-time pseudo-riemannian manifold $\mathcal{V}_4$ and the Minkowski metric $\eta^{\alpha\beta}$ is replaced by a generic pseudo-lorentzian metric $g^{\alpha\beta}$ with the same signature. The stress energy tensor is again conserved, as a consequence of the Einstein equations and of the Bianchi identities, but ordinary derivatives ∂_α are now replaced in (5) by covariant derivatives ∇_α.

3. 5-D RELATIVITY

Let us now introduce a 5-dimensional flat manifold $\mathcal{M}_5$. The signature of cosmological special relativity is $- + + + -$, where the fifth dimension means velocity and the first means time (note that in Carmeli (2002) the opposite signature $+ - - - +$ is adopted, and moreover the fifth dimension means time and the first means velocity). However here we prefer to leave for the moment the possibility for the signature to be $- + + + +$ as well; to this aim we will introduce in the equations a scalar ϵ which can assume the values $+1$ or -1. Let capital latin indices run from 0 to 4. We choose orthonormal coordinates, such that the 5-dimensional line element is:

$$g_{AB}dx^A\,dx^B = \eta_{\alpha\beta}dx^\alpha\,dx^\beta + \epsilon(dx^4)^2 = -dt^2 + dx^2 + dy^2 + dz^2 + \epsilon d\xi^2 \tag{6}$$

where we have denoted $t = x^0$, $x = x^1$, $y = x^2$, $z = x^3$ and $\xi = x^4$ (in the cosmological relativity case we have $\epsilon = -1$ and $\xi = v$).

Let us consider a generic 5-dimensional symmetric 2-tensor $\mathcal{T}^{AB}$.

Then, let us introduce the constant unit vector $\Xi = \partial_{x^4}$ of the fifth dimension; we have $\Xi^A = \delta_4{}^A$, $\Xi_B = \epsilon\delta_B{}^4$ and $\Xi^A\Xi_A = \epsilon$.

The (unique) splitting of $\mathcal{T}^{AB}$ along the ξ-direction and the orthogonal complement of $\mathcal{M}_5$ (which is $\mathcal{M}_4$) is

$$\mathcal{T}^{AB} = T^{AB} + P^A\Xi^B + P^B\Xi^A + E\Xi^A\Xi^B \tag{7}$$

where T^{AB} and P^A are orthogonal to Ξ and T^{AB} is symmetric (it is the ordinary matter-energy tensor).

Let $\mathcal{T}^{AB}$ be conserved in $\mathcal{M}_5$, i.e. let us postulate the following 5-dimensional conservation equation:

$$\partial_A\mathcal{T}^{AB} = 0 \tag{8}$$

This actually is a 5-dimensional generalization of special relativistic continuum dynamics. The splitting of (8) along Ξ and its orthogonal complement gives rise to the following equivalent system

$$\partial_A T^{AB} + \Xi^A\partial_A P^B = 0, \qquad \partial_A P^A + \Xi^A\partial_A E = 0 \tag{9}$$

or, in our coordinates:

$$\partial_\alpha T^{\alpha\beta} + \partial_\xi P^\beta = 0, \qquad \partial_\alpha P^\alpha + \partial_\xi E = 0 \tag{10}$$

Let us now consider instead the case of a generic (non flat) 5-d manifold $\mathcal{V}_5$ in general coordinates, i.e. 5-d general relativity. Let g_{AB} be the 5-d metric tensor, of signature $- + + + -$ or $- + + + +$, depending on the value of ϵ.

Again we have to define the fifth dimension and then to split the 5-d spacetime into the sum of a 4-d spacetime plus the fifth direction. The difference with the case of flat 5-d spacetime is that for a non flat manifold in the general case there are no global coordinates such that:

$$g_{AB}dx^A\,dx^B = g_{\alpha\beta}dx^\alpha\,dx^\beta + \epsilon(d\xi)^2 \tag{11}$$

Let now Ξ be a regular field of unit vectors ($\Xi^A\Xi_A = \epsilon$) with support on $\mathcal{V}_5$ (although this assumption can be relaxed to some domain, subset of $\mathcal{V}_5$), so that the tangent space of any point of $\mathcal{V}_5$ is the sum of the direction of Ξ and of $\mathcal{M}_4$ as the orthogonal complement. Thus Ξ represents the direction of the fifth dimension.

Any tensor index can be projected by means of the projector orthogonal to Ξ: $\delta_A{}^B - \epsilon\;\Xi_A\Xi^B$ and that parallel to Ξ: $\epsilon\;\Xi_A\Xi^B$. Thus any tensor can be splitted into the sum of its pure 4-dimensional space-time component (orthogonal to Ξ), its pure fifth-dimension component (parallel to Ξ), and some mixed components (depending on the order of the tensor). Under this point of view, this is a problem similar to that of splitting of the space-time as the sum of space and time (Jantzen *et al.*, 1992).

Clearly, the definition of the 4-d spacetime is local, unless one additionally supposes that the vector field Ξ globally admits a family of orthogonal 4-d leaf-manifolds. In this case such leaves represent $\mathcal{V}_4$ at any value of the fifth dimension parameter.

If we postulate the following 5-d Einstein equations:

$$\mathcal{G}_{AB} = \chi \mathcal{T}_{AB} \tag{12}$$

where $\mathcal{G}_{AB}$ is the 5-d Einstein tensor, constructed in the usual way by means of the 5-d curvature, we consequently have the conservation of the stress-energy source symmetric tensor $\mathcal{T}^{AB}$, i.e.: $\nabla_A \mathcal{T}^{AB} = 0$. Thus, from decomposition (7) (which still holds at any point of $\mathcal{V}_5$) we have the system:

$$\begin{aligned} &\nabla(\Xi)_A T^{AB} + \epsilon\ \Xi_A \nabla_\Xi T^{AB} + P^A \nabla(\Xi)_A \Xi^B + \nabla_\Xi P^B + \epsilon\ \Xi^B P^A \nabla_\Xi \Xi^A \\ &\qquad + P^B \nabla(\Xi)_A \Xi^A + E \nabla_\Xi \Xi^B = 0 \\ &\nabla(\Xi)_A P^A + \epsilon P^A \nabla_\Xi \Xi_A + \nabla_\Xi E + E \nabla(\Xi)_A \Xi^A = 0 \end{aligned} \tag{13}$$

where $\nabla(\Xi)_A = \nabla_A - \epsilon\ \Xi_A \nabla_\Xi$ and $\nabla_\Xi = \Xi^A \nabla_A$.

The picture is rather more complicated than in the previous case, but things are simpler if we suppose that we can adopt coordinates such that (11) holds, locally or globally. In the latter case, which is the most interesting, the 4-d spacetime leaves are the hypersurfaces of constant ξ, and we still have $\Xi^A = \delta_\xi{}^A$, $\Xi_B = \epsilon \delta_B{}^\xi$, so that we again find (10) but for the replacements of ordinary with covariant derivatives.

Therefore, for the sake of simplicity, in the following we will only refer to the 5-d special relativistic system (10).

4. 5-D HYDRODYNAMICS

Let us suppose that $\mathcal{T}^{AB}$ represents the 5-dimensional stress-energy tensor of a prefect fluid. By analogy with Eq. (2), we may suppose:

$$\mathcal{T}^{AB} = (R + Q) V^A V^B + Q g^{AB} \tag{14}$$

with $R \geq 0$. We don't assume $Q \geq 0$ for cosmological reasons; however the source of particle production we obtain in the following is not directly connected with the sign of Q. As for V^A, we can introduce the following splitting

$$V^A = W^A + \mu \Xi^A \tag{15}$$

with $W^A \Xi_A = 0$; it is natural to suppose $W_A W^A = -W^2 < 0$. For the sake of simplicity we moreover suppose: $\mu > 0$, even if such hypothesis could be avoided.

With our coordinates, we thus have:

$$\begin{aligned} T^{\alpha\beta} &= (R+Q)W^{\alpha}W^{\beta} + Q\eta^{\alpha\beta} \\ P^{\alpha} &= \mu(R+Q)W^{\alpha} \\ E &= \mu^2(R+Q) + \epsilon Q \end{aligned} \tag{16}$$

To have a match between (10) and the ordinary special relativistic system of hydrodynamics (5) we have to suppose that $T^{\alpha\beta}$ is expressed by (2) and that $P^{\alpha} = rU^{\alpha}$. We thus necessarily have:

$$Q = p, \qquad R = \frac{r^2}{\mu^2(\rho+p)} - p, \qquad W^{\alpha} = \frac{\mu(\rho+p)}{r}U^{\alpha} \tag{17}$$

This leaves a degree of freedom for the scalar field μ. In any case for the scalar E we have:

$$E = \frac{r^2}{\rho+p} + \epsilon p. \tag{18}$$

From (10) and (17) we have the differential system:

$$\begin{aligned} \partial_{\alpha}T^{\alpha\beta} + \partial_{\xi}(rU^{\beta}) = 0 \\ \partial_{\alpha}(rU^{\beta}) + \partial_{\xi}E = 0 \end{aligned} \tag{19}$$

The form of E should introduce a correction to the thermodynamic principle of the following kind:

$$dE = (R+p)d\mu^2 + \mu^2 dR + (\mu^2+\epsilon)dp \tag{20}$$

but to the writer it is still unclear how this contribution should fit in (4) and which should be the interpretation of the scalar field μ.

5. COSMOLOGICAL PARTICLE PRODUCTION

Through the years cosmology has become more and more like an experimental science. Recent measurements on magnitude and redshift of supernovae (see e.g. Perlmutter *et al.*, 1999; Riess *et al.*, 1998) seem to indicate that we live in an accelerating universe. It is generally thought that such acceleration is due to some kind of repulsive gravitational force (de Campos, 2002), and that a cosmological perfect fluid with negative pressure can account for it. Negative pressure can be introduced in the stress-energy of a perfect fluid by taking into account the balance of the positive thermodynamic pressure with a negative pressure scalar due to particle creation. We are thus led to the framework of open system cosmology (Prigogine *et al.*, 1989). Alternative cosmologies allow the presence of the cosmological constant in the Einstein equations or of a “quintessential” scalar field (see e.g. Caldwell and Steinhardt, 1998).

In order to take into account particle production one introduces a generic source term ν at right hand side of the matter density conservation equation, which then reads as a balance law (see e.g. Cissoko, 1998)

$$\partial_\alpha(rU^\alpha) = -\nu \tag{21}$$

Let us look at systems (19) and (5); they have the same structure, the only difference coming from the additional terms of (19) containing derivatives with respect to ξ.

We are then able to conclude that the single tensor conservation law (8) unifies the 5 equations of relativistic hydrodynamics (5), provided we get to explain or neglect these additional terms.

A first solution could be to suppose "a priori" that our fields ρ, p, U^α don't depend on the fifth coordinate ξ. This line of thought resembles that followed by Kaluza in his original 5-dimensional theory of gravity, where all derivatives with respect to the fifth coordinate are dropped. We thus exactly obtain the ordinary system (5). But the most interesting possibility is to interpret these terms as effective corrections to relativity.

In cosmological relativity we suppose our variables to be function of velocity as well as of ordinary space and time. Consequently, the additional terms due to the fifth dimension play the role of sources in the equations, which are no more conservation laws. In particular, comparison between Eq. (21) and $(19)_2$, which both allow non-conservation of the number density of matter, leads to

$$\nu = \partial_\xi E \tag{22}$$

This identification seems to suggest that phenomena of particle creation can be given a geometrical interpretation: they are originated by the 5-dimensional structure of the space-time. In the space-velocity-time of cosmological relativity, since $\xi = v$, we have the following expression for the cosmological source of particle production:

$$\nu = 2\mu(R + p)\left(\frac{d\mu}{dv}\right) + \phi^2\left(\frac{dR}{dv}\right) + (\mu^2 - 1)\left(\frac{dp}{dv}\right) \tag{23}$$

even if at the moment we still have no elements to predict the functional dependence of the thermodynamical variables and of the (say quintessential) scalar field μ on the receding velocity v.

In our framework, ordinary special relativity and ordinary cosmological relativity appear to be approximations of a 5-dimensional theory, whose additional terms are usually negligible here and today, due to some weak coupling, but may have played a role in the inflationary expansion of the universe. Here, due to $(19)_1$, even energy is not exactly conserved.

Equation (19) actually show that effects due to the fifth dimension are not negligible when particle production phenomena are present; they moreover suggest that such phenomena can always be interpreted as effects due to the fifth dimension.

Conversely, in the 5-d general relativistic case, the 5-d Einstein equations also suggest that particle production phenomena have a direct influence on the geometry of the spacetime.

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